Available online at www.sciencedirect.com
ScienceDirect
Journal of Geometry and Physics 57 (2007) 833–840
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

A pinching theorem for the normal scalar curvature of invariant submanifolds

Franki Dillen*, Johan Fastenakels, Joeri Van der Veken

Katholieke Universiteit Leuven, Departement Wiskunde, Celestijnenlaan 200 B, B-3001 Leuven, Belgium

Received 8 March 2006; accepted 9 June 2006
Available online 4 August 2006

Abstract

We prove some inequalities relating intrinsic and extrinsic curvature invariants for invariant submanifolds of Kaehlerian and Sasakian space forms. When we restrict to invariant submanifolds of odd-dimensional unit spheres or invariant submanifolds of complex Euclidean space, one of the inequalities gives a positive answer to a conjecture, proposed in [P.J. De Smet, F. Dillen, L. Verstraelen, L. Vrancken, A pointwise inequality in submanifold theory, Arch. Math. (Brno) 35 (1999) 115–128].

MSC: primary 53B25; 53B35; secondary 53A07

Keywords: Kaehlerian space form; Sasakian space form; Invariant submanifold; Scalar curvature; Inequality

1. Preliminaries

Let M^n be a Riemannian manifold of dimension n with Levi-Civita connection ∇ and Riemann–Christoffel curvature tensor R. If $\{e_1, \ldots, e_n\}$ is an orthonormal basis for T_pM, then we define, following B.Y. Chen's convention, the *scalar curvature* of M^n at p by

$$\tau = \sum_{i<j=1}^{n} \langle R(e_i, e_j)e_j, e_i\rangle \tag{1}$$

and the *normalized scalar curvature* of M^n at p by $\rho = \frac{2}{n(n-1)}\tau$.

Now let $\widetilde{M}^m$ be another Riemannian manifold of dimension $m > n$ with Levi-Civita connection $\widetilde{\nabla}$ and Riemann–Christoffel curvature tensor $\widetilde{R}$ and let $f : M^n \to \widetilde{M}^m$ be an isometric immersion. Then we have the formulas of Gauss and Weingarten, which state that for vector fields X and Y tangent to M^n and for a normal vector field U it holds that

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2}$$

* Corresponding author. Tel.: +32 16327004; fax: +32 16327998.
E-mail addresses: franki.dillen@wis.kuleuven.be (F. Dillen), johan.fastenakels@wis.kuleuven.be (J. Fastenakels), joeri.vanderveken@wis.kuleuven.be (J. Van der Veken).

0393-0440/$ - see front matter
doi:10.1016/j.geomphys.2006.06.006

$$\widetilde{\nabla}_X U = -A_U X + \nabla^{\perp}_X U, \tag{3}$$

where h is a symmetric (1, 2)-tensor field, taking values in the normal bundle, called the *second fundamental form*, and A_U is a symmetric (1, 1)-tensor field, called the *shape operator* associated with U. $\nabla^{\perp}$ is a connection in the normal bundle and we denote its curvature tensor by $R^{\perp}$. The equations of Gauss and Ricci are then given by

$$\langle R(X, Y)Z, T\rangle = \langle \widetilde{R}(X, Y)Z, T\rangle + \langle h(X, T), h(Y, Z)\rangle - \langle h(X, Z), h(Y, T)\rangle, \tag{4}$$

$$\langle R^{\perp}(X, Y)U, V\rangle = \langle \widetilde{R}(X, Y)U, V\rangle + \langle [A_U, A_V]X, Y\rangle, \tag{5}$$

for tangent vectors X, Y, Z and T and normal vectors U and V. We define the *normal scalar curvature* of M^n at p by

$$\tau^{\perp} = \sqrt{\sum_{i<j=1}^{n} \sum_{\alpha<\beta=1}^{m-n} \langle R^{\perp}(e_i, e_j)u_\alpha, u_\beta\rangle^2} \tag{6}$$

and the normalized normal scalar curvature of M^n at p by $\rho^{\perp} = \frac{2}{n(n-1)}\tau^{\perp}$, which corresponds to the definition proposed in [5]. Here $\{e_1, \ldots, e_n\}$ is as above and $\{u_1, \ldots, u_{m-n}\}$ is an orthonormal basis for $T_p^{\perp}M$. Another extrinsic curvature invariant that we will use is

$$\|h\|^2 = \sum_{i,j=1}^{n} \|h(e_i, e_j)\|^2 = \sum_{\alpha=1}^{m-n} \|A_{u_\alpha}\|^2. \tag{7}$$

Denoting by $H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i)$ the mean curvature vector of the submanifold at p, the following conjecture was formulated in [5]:

Conjecture 1. *Let $f : M^n \to \widetilde{M}^m(c)$ be an isometric immersion, where $\widetilde{M}^m(c)$ is a real space form of constant sectional curvature c. Then*

$$\rho \leq \|H\|^2 - \rho^{\perp} + c.$$

This is an extension of the well-known inequality (see for instance [3])

$$\rho \leq \|H\|^2 + c. \tag{8}$$

Let $\widetilde{M}^{2m}(c)$ be a *Kaehlerian space form* of real dimension $2m$ (complex dimension m) of constant holomorphic sectional curvature c. The curvature tensor $\widetilde{R}$ of a Kaehlerian space form is given by

$$\widetilde{R}(X, Y)Z = \frac{c}{4}(\langle Y, Z\rangle X - \langle X, Z\rangle Y + \langle JY, Z\rangle JX - \langle JX, Z\rangle JY - 2\langle JX, Y\rangle JZ), \tag{9}$$

where J denotes the complex structure of $\widetilde{M}^{2m}(c)$. If this space is complete and simply connected, it is well known that it is isometric to

- a complex projective space $\mathbb{C}P^m(c)$, if $c > 0$;
- the complex Euclidean space $\mathbb{C}^m$, if $c = 0$;
- a complex hyperbolic space $\mathbb{C}H^m(c)$, if $c < 0$.

We say that a $2n$-dimensional submanifold $f : M^{2n} \to \widetilde{M}^{2m}(c)$ is *invariant* (also called *complex* or *Kaehler*) if $J(T_pM) \subset T_pM$ for every $p \in M^{2n}$. For an invariant submanifold, one has

$$A_{JU}X = -A_U JX = JA_U X. \tag{10}$$

A Sasakian manifold can be seen as an odd-dimensional analogue of a Kaehlerian manifold. We call an odd-dimensional Riemannian manifold, with Levi-Civita connection ∇, *Sasakian* if it carries a unit vector field ξ, a one-form η and a (1,1)-tensor field ϕ such that for all tangent vectors X and Y

$$\phi^2(X) = -X + \eta(X)\xi; \qquad \eta(X) = \langle X, \xi\rangle; \qquad \langle \phi X, \phi Y\rangle = \langle X, Y\rangle - \eta(X)\eta(Y); \tag{11}$$

$$(\nabla_X \phi)Y = \langle X, Y\rangle\xi - \eta(Y)X. \tag{12}$$

If only the equations (11) are satisfied, the manifold is said to carry an *almost contact metric structure*. Using these equations one easily proves

$$\eta(\xi) = 1; \qquad \phi(\xi) = 0; \qquad \eta \circ \phi = 0. \tag{13}$$

A *Sasakian space form* $\widetilde{M}^{2m+1}(c)$ of constant ϕ-holomorphic sectional curvature c is a Riemannian manifold with Sasakian structure (ξ, η, ϕ), such that all 2-planes spanned by a basis of the form $\{X, \phi X\}$ have the same sectional curvature. The curvature tensor $\widetilde{R}$ of these spaces takes the form

$$\begin{aligned} \widetilde{R}(X,Y)Z = {} & \frac{c+3}{4}(\langle Y,Z\rangle X - \langle X,Z\rangle Y) \\ & + \frac{c-1}{4}(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \langle X,Z\rangle\eta(Y)\xi - \langle Y,Z\rangle\eta(X)\xi \\ & + \langle\phi Y,Z\rangle\phi X - \langle\phi X,Z\rangle\phi Y + 2\langle X,\phi Y\rangle\phi Z). \end{aligned} \tag{14}$$

If a Sasakian space form $\widetilde{M}^{2m+1}(c)$ is complete and simply connected, it is isometric to one of the following spaces:

- a D-homothetic transformation of $S^{2m+1}(1)$, if $c > -3$ (which coincides with $S^{2m+1}(1)$ for $c = 1$);
- the real number space with standard Sasakian structure $\mathbb{R}^{2m+1}(-3)$, if $c = -3$;
- a line bundle over a complex hyperbolic space $(\mathbb{R}, \mathbb{C}H^m)$, if $c < -3$.

Now let $f : M^{2n+1} \to \widetilde{M}^{2m+1}(c)$ be an isometric immersion of a Riemannian manifold into a Sasakian space form. By analogy with the Kaehlerian case, we say that the submanifold is *invariant* if $\phi T_pM \subset T_pM$ for every $p \in M^{2n+1}$. In that case, ξ is tangent to M. For invariant submanifolds, one has

$$A_{\phi U}X = -A_U\phi X = \phi A_U X; \qquad A_U\xi = 0. \tag{15}$$

We call a Sasakian manifold $\widetilde{M}^{2m+1}$ *regular* if every orbit of ξ is a closed set. In this case the set of all these orbits forms a differentiable manifold and we can construct a Riemannian submersion $\pi : \widetilde{M}^{2m+1} \to \overline{\widetilde{M}} = \widetilde{M}^{2m+1}/\xi$. Locally the fibration π always exists. The base space carries the structure of a Kaehlerian manifold of real dimension $2m$: if we denote, for a vector field X on $\overline{\widetilde{M}}^{2m}$, by X^* its horizontal lift, i.e. the unique vector field on $\widetilde{M}^{2m+1}$ orthogonal to the fibres of π and such that $\pi_*|_p(X^*|_p) = X|_{\pi(p)}$ for all $p \in \widetilde{M}^{2m+1}$, then the complex structure of $\overline{\widetilde{M}}^{2m}$ is given by $(JX)^* = \phi X^*$. Moreover, if $\widetilde{M}^{2m+1}$ is a Sasakian space form of constant ϕ-holomorphic sectional curvature c, then the base space is a Kaehlerian space form of constant holomorphic sectional curvature $c+3$. From now on, we assume we are in this case. If M^{2n+1} is an invariant submanifold of $\widetilde{M}^{2m+1}(c)$, then one can prove that the image $\overline{M}^{2n}$ of M^{2n+1} under π is an invariant submanifold of $\overline{\widetilde{M}}^{2m}(c+3)$. So we have the following commutative diagram:

$$\begin{array}{ccc} M^{2n+1}, \nabla & \xrightarrow[A,\,h,\,\nabla^\perp]{f} & \widetilde{M}^{2m+1}(c), \widetilde{\nabla} \\ \Big\downarrow{\scriptstyle \pi|_M} & & \Big\downarrow{\scriptstyle \pi} \\ \overline{M}^{2n}, \overline{\nabla} & \xrightarrow[\overline{A},\,\overline{h},\,\overline{\nabla}^\perp]{\overline{f}} & \overline{\widetilde{M}}^{2m}(c+3), \overline{\widetilde{\nabla}}. \end{array}$$

For X, Y tangent to $\overline{M}^{2n}$ and U normal to $\overline{M}^{2n}$, we have

$$(\overline{\nabla}_X Y)^* = \nabla_{X^*}Y^* - \langle\phi X^*, Y^*\rangle\xi \tag{16}$$

$$(\overline{h}(X,Y))^* = h(X^*, Y^*) \tag{17}$$

$$(\overline{A}_U X)^* = A_{U^*}X^* \tag{18}$$

$$(\overline{\nabla}^\perp_X U)^* = \nabla^\perp_{X^*}U^*. \tag{19}$$

For a rigorous treatment of this theory, see [12,2].

2. Invariant submanifolds of Kaehler manifolds

In this section, we consider invariant submanifolds of Kaehlerian space forms. We use the same notation as above. Note that we can choose an orthonormal tangent frame of the form $\{e_1, \ldots, e_n, e_{n+1} = Je_1, \ldots, e_{2n} = Je_n\}$ and an orthonormal normal frame of the form $\{u_1, \ldots, u_{m-n}, u_{m-n+1} = Ju_1, \ldots, u_{2(m-n)} = Ju_{m-n}\}$. The following lemma was proved in [8]:

Lemma 1. *Let M^{2n} be an invariant submanifold of a Kaehlerian space form $\widetilde{M}^{2m}(c)$ of constant holomorphic sectional curvature c. Then*

$$\frac{1}{n}\|h\|^4 \leq \sum_{\alpha,\beta=1}^{2(m-n)} \|[A_{u_\alpha}, A_{u_\beta}]\|^2 \leq \|h\|^4.$$

Studying the proof, one sees that equality holds in the second inequality if and only if the complex rank of $\mathcal{A} = \sum_{\alpha=1}^{2(m-n)} A_{u_\alpha}^2$ is at most 1 and that equality holds in the first inequality if and only if M^{2n} is *Einstein*, i.e. the Ricci curvature tensor $S(X, Y) = \sum_{i=1}^{2n} \langle R(X, e_i)e_i, Y\rangle$ is a scalar multiple of the metric at every point. In the case that M^{2n} is a complex Einstein hypersurface, we have, see for instance [10], that either M^{2n} is totally geodesic or that $c > 0$ and the Ricci curvature tensor satisfies

$$S(X, Y) = \frac{n}{2}c\langle X, Y\rangle. \tag{20}$$

We now prove the following:

Theorem 1. *Let M^{2n} be an invariant submanifold of a Kaehlerian space form $\widetilde{M}^{2m}(c)$ of constant holomorphic sectional curvature c. Then*

(i) $4n(\tau^\perp)^2 \geq (n(n+2)c - 2\tau)^2 + c^2n^2(m - n - 1)$,
(ii) $4(\tau^\perp)^2 \leq \left((n^2 + n + 1)c - 2\tau\right)^2 + c^2(mn - n^2 - 1)$,

with equality in (i) *if and only if M^{2n} is Einstein and equality in* (ii) *if and only if the complex rank of $\mathcal{A} = \sum_{\alpha=1}^{2(m-n)} A_{u_\alpha}^2$ is at most 1.*

Proof. Using Gauss's equation (4), the expression for $\widetilde{R}$ (9) and the definition of the scalar curvature (1) one can compute

$$\tau = \sum_{i<j=1}^{2n} \langle R(e_i, e_j)e_j, e_i\rangle = \frac{1}{2}n(n+1)c - \frac{1}{2}\|h\|^2. \tag{21}$$

The normal scalar curvature of M^{2n} is given by

$$(\tau^\perp)^2 = \sum_{\alpha<\beta=1}^{2(m-n)} \sum_{i<j=1}^{2n} \langle R^\perp(e_i, e_j)u_\alpha, u_\beta\rangle^2$$

and thus, using Ricci's equation,

$$\begin{aligned}(\tau^\perp)^2 &= \sum_{\alpha<\beta=1}^{2(m-n)} \sum_{i<j=1}^{2n} \left(\langle [A_{u_\alpha}, A_{u_\beta}]e_i, e_j\rangle + \frac{c}{2}\langle e_i, Je_j\rangle\langle Ju_\alpha, u_\beta\rangle\right)^2 \\ &= \sum_{\alpha<\beta=1}^{2(m-n)} \sum_{i<j=1}^{2n} \langle [A_{u_\alpha}, A_{u_\beta}]e_i, e_j\rangle^2 + \frac{c^2}{4} \sum_{\alpha<\beta=1}^{2(m-n)} \sum_{i<j=1}^{2n} \langle e_i, Je_j\rangle^2\langle Ju_\alpha, u_\beta\rangle^2 \\ &\quad + c \sum_{\alpha<\beta=1}^{2(m-n)} \sum_{i<j=1}^{2n} \langle e_i, Je_j\rangle\langle Ju_\alpha, u_\beta\rangle\langle [A_{u_\alpha}, A_{u_\beta}]e_i, e_j\rangle\end{aligned}$$

$$= \frac{1}{4}\sum_{\alpha,\beta=1}^{2(m-n)}\sum_{i,j=1}^{2n}\langle [A_{u_\alpha}, A_{u_\beta}]e_i, e_j\rangle^2 + \frac{c^2}{4}n(m-n) - c\sum_{\alpha=1}^{m-n}\sum_{i=1}^{n}\langle [A_{u_\alpha}, A_{Ju_\alpha}]e_i, Je_i\rangle$$
$$= \frac{1}{4}\sum_{\alpha,\beta=1}^{2(m-n)}\|[A_{u_\alpha}, A_{u_\beta}]\|^2 + \frac{c^2}{4}n(m-n) + 2c\sum_{\alpha=1}^{m-n}\sum_{i=1}^{n}\|A_{u_\alpha}e_i\|^2$$
$$= \frac{1}{4}\sum_{\alpha,\beta=1}^{2(m-n)}\|[A_{u_\alpha}, A_{u_\beta}]\|^2 + \frac{c^2}{4}n(m-n) + \frac{c}{2}\|h\|^2.$$

Now using the first inequality in Lemma 1, we obtain

$$\begin{aligned}(\tau^\perp)^2 &\geq \frac{1}{4n}\|h\|^4 + \frac{c}{2}\|h\|^2 + \frac{c^2}{4}n(m-n)\\ &= \frac{1}{4n}(\|h\|^2 + nc)^2 + \frac{c^2}{4}n(m-n-1)\\ &= \frac{1}{4n}(n(n+2)c - 2\tau)^2 + \frac{c^2}{4}n(m-n-1),\end{aligned}$$

with equality holding if and only if M^{2n} is Einstein. This proves (i). On the other hand, using the second inequality in Lemma 1 yields

$$\begin{aligned}(\tau^\perp)^2 &\leq \frac{1}{4}\|h\|^4 + \frac{c}{2}\|h\|^2 + \frac{c^2}{4}n(m-n)\\ &= \frac{1}{4}(\|h\|^2 + c)^2 + \frac{c^2}{4}(mn - n^2 - 1)\\ &= \frac{1}{4}((n^2+n+1)c - 2\tau)^2 + \frac{c^2}{4}(mn - n^2 - 1),\end{aligned}$$

with equality if and only if the complex rank of $\mathcal{A}$, defined above, is less than or equal to 1. This finishes the proof. □

We can give some more details on the submanifolds realizing the equality in (ii). First, notice that after a suitable choice of a normal basis, the shape operators are of the form

$$A_{u_1} = \left(\begin{array}{ccc|ccc} \lambda & 0 & & & & \\ 0 & 0 & & & 0 & \\ & & \ddots & & & \\ \hline & & & -\lambda & 0 & \\ & 0 & & 0 & 0 & \\ & & & & & \ddots \end{array}\right) \quad \text{and} \quad A_{Ju_1} = \left(\begin{array}{ccc|ccc} & & & \lambda & 0 & \\ & 0 & & 0 & 0 & \\ & & & & & \ddots \\ \hline \lambda & 0 & & & & \\ 0 & 0 & & & 0 & \\ & & \ddots & & & \end{array}\right),$$

with $\lambda = \|h(e_1, e_1)\|$, and all others are 0. If we suppose the submanifolds to be complete, then we have the following [1]: if $c = 0$, M^{2n} is totally geodesic, or a complex curve or a complex cylinder in the sense of Abe. If $c > 0$, M^{2n} is totally geodesic or a complex curve. As far as we know there is no such result known in the complex hyperbolic case.

We can now prove a special case of Conjecture 1:

Corollary 1. *For an invariant submanifold M^{2n} of $\mathbb{C}^m \cong \mathbb{E}^{2m}$, we have*

$$\rho \leq -\rho^\perp.$$

Proof. Taking $c = 0$ in the second inequality of Theorem 1, we get $\tau^\perp \leq |\tau|$, or, on multiplying by $\frac{2}{2n(2n-1)}$, $\rho^\perp \leq |\rho|$. Because an invariant submanifold is automatically minimal, (8) implies that $\rho \leq 0$, and hence we obtain the result. □

The following result was proven in a different way in [4].

Corollary 2. *If an invariant submanifold M^{2n} of a Kaehlerian space form $\widetilde{M}^{2m}(c)$ is normally flat, then $c = 0$ and M^{2n} is totally geodesic.*

Proof. Taking $\tau^\perp = 0$ in the first inequality in Theorem 1 yields that we must have equality, and hence M^{2n} is Einstein and we are in one of the following cases:

- $c = \tau = 0$, and thus $\|h\|^2 = 0$;
- $m = n + 1$ and $\tau = n(n+2)c$, and thus $\|h\|^2 = -nc$.

Because all invariant Einstein submanifolds of $\mathbb{C}H^m$ are totally geodesic (see [11]), we find that the second case cannot occur, except when $c = 0$ and then it coincides with the first case. □

3. Invariant submanifolds of Sasakian manifolds

In this section we will consider invariant submanifolds of Sasakian space forms, using the Riemannian submersion π, introduced in Section 1. We use the same notation here. Let $\{e_1, \ldots, e_n, e_{n+1} = Je_1, \ldots, e_{2n} = Je_n\}$ be an orthonormal frame on $\overline{M}^{2n}$; then $\{e_1^*, \ldots, e_{2n}^*, \xi\}$ is an orthonormal frame on M^{2n+1}. Similarly, if $\{u_1, \ldots, u_{m-n}, u_{m-n+1} = Ju_1, \ldots, u_{2(m-n)} = Ju_{m-n}\}$ is an orthonormal frame normal to $\overline{M}^{2n}$ in $\overline{M}^{2m}(c+3)$, then $\{u_1^*, \ldots, u_{2(m-n)}^*\}$ is an orthonormal frame normal to M^{2n+1} in $\widetilde{M}^{2m+1}(c)$. The following lemma was proved in [9].

Lemma 2. *Let M^{2n+1} be an invariant submanifold of a Sasakian space form $\widetilde{M}^{2m+1}(c)$ of constant ϕ-holomorphic sectional curvature c. Then*

$$\frac{1}{n}\|h\|^4 \leq \sum_{\alpha,\beta=1}^{2(m-n)} \|[A_{u_\alpha^*}, A_{u_\beta^*}]\|^2 \leq \|h\|^4.$$

From the proof, it follows that equality holds in the second inequality if and only if the rank of $\mathcal{A} = \sum_{\alpha=1}^{2(m-n)} A_{u_\alpha^*}^2$ is less than or equal to 2 and that equality holds in the first inequality if and only if M^{2n+1} is *η-Einstein*, meaning that the Ricci curvature tensor is of the form $S(X,Y) = a\langle X,Y\rangle + b\eta(X)\eta(Y)$ for some functions a and b. Remark that M^{2n+1} is η-Einstein if and only if $\overline{M}^{2n}$ is Einstein.

Lemma 3. *Denote by $\tau^\perp$ the normal scalar curvature of M^{2n+1} in $\widetilde{M}^{2m+1}(c)$ and by $\overline{\tau}^\perp$ the normal scalar curvature of $\overline{M}^{2n}$ in $\overline{\widetilde{M}}^{2m}(c+3)$. Then*

$$(\overline{\tau}^\perp)^2 = (\tau^\perp)^2 + 2\|h\|^2 + 2(c+1)n(m-n).$$

Proof. For X, Y tangent to $\overline{M}^{2n}$ and U, V normal to $\overline{M}^{2n}$, one can verify straightforwardly, using (16), (19) and the equalities $[\xi, U^*] = 0$ and $\widetilde{\nabla}_X \xi = -\phi X$ (the last one following from (12)), that

$$\langle R^\perp(X^*, Y^*)U^*, V^*\rangle = \langle \overline{R}^\perp(X,Y)U, V\rangle + 2\langle JX, Y\rangle\langle JU, V\rangle.$$

This implies

$$\begin{aligned}
(\overline{\tau}^\perp)^2 &= \sum_{\alpha<\beta=1}^{2(m-n)} \sum_{i<j=1}^{2n} \langle \overline{R}^\perp(e_i, e_j)u_\alpha, u_\beta\rangle^2 \\
&= \sum_{\alpha<\beta=1}^{2(m-n)} \sum_{i<j=1}^{2n} (\langle R^\perp(e_i^*, e_j^*)u_\alpha^*, u_\beta^*\rangle - 2\langle Je_i, e_j\rangle\langle Ju_\alpha, u_\beta\rangle)^2 \\
&= (\tau^\perp)^2 - \sum_{\alpha<\beta=1}^{2(m-n)} \sum_{i=1}^{2n} \langle R^\perp(e_i^*, \xi)u_\alpha^*, u_\beta^*\rangle^2 - 4\sum_{\alpha=1}^{m-n}\sum_{i=1}^{n} \langle R^\perp(e_i^*, (Je_i)^*)u_\alpha^*, (Ju_\beta)^*\rangle + 4n(m-n)
\end{aligned}$$

$$= (\tau^\perp)^2 - 4\sum_{\alpha=1}^{m-n}\sum_{i=1}^{n}\left(\langle [A_{u_\alpha^*}, A_{\phi u_\alpha^*}]e_i^*, \phi e_i^*\rangle - \frac{c-1}{2}\right) + 4n(m-n)$$
$$= (\tau^\perp)^2 + 8\sum_{\alpha=1}^{m-n}\sum_{i=1}^{n}\|A_{u_\alpha^*}e_i^*\|^2 + 4\frac{c-1}{2}n(m-n) + 4n(m-n)$$
$$= (\tau^\perp)^2 + 2\|h\|^2 + 2(c+1)n(m-n). \quad \square$$

Theorem 2. *Let M^{2n+1} be an invariant submanifold of a Sasakian space form $\widetilde{M}^{2m+1}(c)$ of constant ϕ-holomorphic sectional curvature c. Then*

(i) $4n(\tau^\perp)^2 \geq (n(n+2)c + 3n^2 - 2\tau)^2 + (c-1)^2n^2(m-n-1)$;
(ii) $4(\tau^\perp)^2 \leq ((n^2+n+1)c + (3n^2+n-1) - 2\tau)^2 + (c-1)^2(mn-n^2-1)$;

with equality in (i) *if and only if M^{2n+1} is η-Einstein and equality in* (ii) *if and only if the rank of $\mathcal{A} = \sum_{\alpha=1}^{2(m-n)} A_{u_\alpha^*}^2$ is at most 2.*

Proof. Using Gauss's equation (4), the expression for $\widetilde{R}$ (14) and the definition of scalar curvature (1), we obtain

$$\tau = \frac{1}{2}n(n+1)c + \frac{1}{2}n(3n+1) - \frac{1}{2}\|h\|^2. \tag{22}$$

Remark that $|h| = |\overline{h}|$, due to (17). From Theorem 1 we then have

$$(\overline{\tau}^\perp)^2 \geq \frac{1}{4n}\|h\|^4 + \frac{c+3}{2}\|h\|^2 + \frac{(c+3)^2}{4}n(m-n)$$

and hence by the previous lemma

$$(\tau^\perp)^2 \geq \frac{1}{4n}\|h\|^4 + \frac{c-1}{2}\|h\|^2 + \frac{(c-1)^2}{4}n(m-n).$$

The inequality (i) now follows from (22). In an analogous way one can prove (ii). The remaining statements follow from the observations that M^{2n+1} is η-Einstein if and only if $\overline{M}^{2n}$ is Einstein and that the rank of $\mathcal{A}$ equals the real rank of $\overline{\mathcal{A}} = \sum_{\alpha=1}^{m-n} \overline{A}_{u_\alpha}$. $\square$

Again, we can prove a special case of Conjecture 1:

Corollary 3. *Let M^{2n+1} be a submanifold of $S^{2m+1}(1)$ which is invariant with respect to the standard Sasakian structure on the unit sphere. Then*

$$\rho \leq 1 - \rho^\perp.$$

Proof. Taking $c = 1$ in the second inequality of Theorem 2 yields

$$(\tau^\perp)^2 \leq (n(2n+1) - \tau)^2.$$

Multiplying by $\left(\frac{2}{(2n+1)2n}\right)^2$ gives $(\rho^\perp)^2 \leq (1-\rho)^2$ and because $1 - \rho \geq 0$, due to (8), we get the result. $\square$

The following corollary was proven in a different way in [6].

Corollary 4. *If an invariant submanifold M^{2n+1} of a Sasakian space form $\widetilde{M}^{2m+1}(c)$ is normally flat, then there are two possibilities:*

- *$c = 1$ and M^{2n+1} is totally geodesic;*
- *$c = -1$, $m = n+1$, $\tau = n(n-1)$ and M^{2m-1} is η-Einstein.*

Proof. If $\tau^\perp = 0$, we must have equality in the first inequality of Theorem 2. Hence M^{2n+1} is η-Einstein and we have the following possibilities:

- $c = 1$ and $\tau = \frac{1}{2}n(n+2)c + \frac{3}{2}n^2$, and thus $h = 0$;
- $m = n + 1$ and $\tau = \frac{1}{2}n(n+2)c + \frac{3}{2}n^2$, and thus $\|h\|^2 = -(c-1)n$.

But since M^{2n+1} is η-Einstein in $\widetilde{M}^{2m+1}(c)$, $\overline{M}^{2n}$ is Einstein in $\overline{\widetilde{M}}^{2m}(c+3)$ and according to (20) this implies in the second case that either $\overline{M}^{2(m-1)}$ is totally geodesic, and then we are in the first case, or that $c > 0$ and $\overline{\tau} = \frac{1}{2}n^2(c+3)$, and thus $\|h\|^2 = n(c+3)$. So we find that in the second case $c = -1$, $\tau = n(n-1)$ and $\|h\|^2 = 2n$. □

To show that the second case in the previous corollary actually occurs, we give the following example. Consider the complex quadric in $\mathbb{C}P^m(2)$ with equation $z_0^2 + z_1^2 + \cdots + z_m^2 = 0$. It is well known that Q^{m-1} is Einstein and hence its lift is an η-Einstein submanifold of $\widetilde{M}^{2m+1}(-1)$. Moreover $\|h\|^2 = \|\overline{h}\|^2 = 2n$ from (21) and (20). Moreover we see from [7] that every submanifold from the second case is locally isometric to this example.

Acknowledgement

The second author is a Research Assistant of the Research Foundation—Flanders (FWO).

References

[1] K. Abe, Applications of a Riccati type differential equation to Riemannian manifolds with totally geodesic distributions, Tôhoku Math. J. 25 (1973) 425–444.
[2] D.E. Blair, Riemannian Geometry of Contact and Symplectic Manifolds, in: Progr. Math., vol. 203, Birkhäuser, Basel, 2002.
[3] B.Y. Chen, Mean curvature and shape operator of isometric immersions in real-space-forms, Glasg. Math. J. 38 (1996) 87–97.
[4] B.Y. Chen, K. Ogiue, Some extrinsic results for Kaehler submanifolds, Tamkang J. Math. 4 (2) (1973) 207–213.
[5] P.J. De Smet, F. Dillen, L. Verstraelen, L. Vrancken, A pointwise inequality in submanifold theory, Arch. Math. (Brno) 35 (1999) 115–128.
[6] K. Kenmotsu, On Sasakian immersions, in: Sem. on Contact Manifolds, Pub. of the Study Group of Geometry, vol. 4, 1970, pp. 42–59.
[7] K. Kenmotsu, Local classification of invariant η-Einstein submanifolds of codimension 2 in a Sasakian manifold with constant ϕ-sectional curvature, Tôhoku Math. J. 22 (1970) 270–272.
[8] M. Kon, On some complex submanifolds in Kaehler manifolds, Canad. J. Math. 26 (6) (1974) 1442–1449.
[9] M. Kon, Invariant submanifolds of Sasakian manifolds, Math. Ann. 219 (1976) 277–290.
[10] K. Nomizu, B. Smyth, Differential geometry of complex hypersurfaces II, J. Math. Soc. Japan 20 (1968) 499–521.
[11] M. Umehara, Einstein Kaehler submanifolds of a complex linear or hyperbolic space, Tôhoku Math. J. 39 (1987) 385–389.
[12] K. Yano, S. Ishihara, Invariant submanifolds of an almost contact manifold, Kodai Math. Sem. Rep. 21 (1969) 350–364.